

θ -Summation and Hardy Spaces¹

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A general summability method of Fourier series and Fourier transforms is given
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θ -means of a distribution is bounded from $H_p(\mathbf{T})$ to $L_p(\mathbf{T})$ ($p_0 < p < \infty$) and is of weak type (1,1), where $H_p(\mathbf{T})$ is the classical Hardy space and $p_0 < 1$ is depending only on θ . As a consequence we obtain that the θ -means of a function $f \in L_1(\mathbf{T})$ converge a.e. to f . For the endpoint p_0 we get that the maximal operator is of weak type $(H_{p_0}(\mathbf{T}), L_{p_0}(\mathbf{T}))$. Moreover, we prove that the θ -means are uniformly bounded on the spaces $H_p(\mathbf{T})$ whenever $p_0 < p < \infty$ and are uniformly of weak type $(H_{p_0}(\mathbf{T}), H_{p_0}(\mathbf{T}))$. Thus, in the case $f \in H_p(\mathbf{T})$, the θ -means converge to f in $H_p(\mathbf{T})$ norm ($p_0 < p < \infty$). The same results are proved for the conjugate θ -means and for Fourier transforms, too. Some special cases of the θ -summation are considered, such as the Weierstrass, Picar, Bessel, Fejér, Riemann, de La Vallée-Poussin, Rogosinski and Riesz summations. © 2000 Academic Press

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1. INTRODUCTION

The Hardy–Lorentz spaces $H_{p,q}(\mathbf{T})$ of distributions are introduced with the $L_{p,q}(\mathbf{T})$ Lorentz norm of the non-tangential maximal function. Of course, $H_p(\mathbf{T}) = H_{p,p}(\mathbf{T})$ are the usual Hardy spaces ($0 < p \leq \infty$).

Butzer and Nessel [3] and recently Bokor, Schipp, Szili and Vértesi [2, 11, 12, 16, 17] considered a general method of summation, the so-called θ -summability. The θ -means of Fourier transforms can be written in a

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natural way as a singular integral of the Fourier transform of θ , $\hat{\theta}$ (see Butzer and Nessel [3]). They proved that if $\hat{\theta}$ can be estimated by a non-increasing integrable function, then the θ -means of a function $f \in L_1(\mathbf{R})$ converge a.e. to f . This convergence result is also proved there for the θ -means of Fourier series. As special cases they considered the Weierstrass, Picar, Bessel, Fejér, de La Vallée-Poussin and Riesz summations. For example, they verified that the Riesz means $\sigma_T^{\alpha, \gamma} f$ converge a.e. to f as $T \rightarrow \infty$ if $f \in L_1(\mathbf{R})$ and $\gamma = 1, 2$ (see also Stein and Weiss [14]).

The author [21] generalized this last result and proved that the maximal Riesz operator $\sigma_*^{\alpha, \gamma} := \sup_{T>0} |\sigma_T^{\alpha, \gamma}|$ is bounded from $H_p(\mathbf{R})$ to $L_p(\mathbf{R})$ provided that $0 < \alpha < \infty$, $1 \leq \gamma < \infty$, $1/(\min(\alpha, 1) + 1) < p < \infty$ and, moreover, it is of weak type $(1, 1)$, i.e.

$$\sup_{\rho>0} \rho \lambda(\sigma_*^{\alpha, \gamma} f > \rho) \leq C \|f\|_1 \quad (f \in L_1(\mathbf{R}))$$

(this last result for $\alpha = \gamma = 1$ can also be found in Zygmund [23] and Móricz [10]). This weak type inequality assures already the a.e. convergence of the Riesz means mentioned above.

In this paper we generalize these results. First we consider the θ -means of Fourier series and prove that the θ -means $U_n^\theta f$ of a function $f \in L_1(\mathbf{T})$ can be written also as a singular integral of f and $\hat{\theta}$ over \mathbf{R} . We introduce the maximal operator $U_*^\theta := \sup_{n \in \mathbf{N}} |U_n^\theta|$, the conjugate distribution \tilde{f} , the conjugate θ -means $\tilde{U}_n^\theta f$ and the conjugate maximal operator \tilde{U}_*^θ .

Under some weak conditions on θ and $\hat{\theta}$ we will show that the maximal operators U_*^θ and \tilde{U}_*^θ are bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ whenever $p_0 < p < \infty$, $0 < q \leq \infty$ and are of weak type $(1, 1)$. The parameter p_0 is less than 1 and depending on θ . For this endpoint we can verify that the preceding two maximal operators are of weak type $(H_{p_0}(\mathbf{T}), L_{p_0}(\mathbf{T}))$.

A usual density argument implies then that $U_n^\theta f \rightarrow f$ a.e. and $\tilde{U}_n^\theta f \rightarrow \tilde{f}$ a.e. as $n \rightarrow \infty$, provided that $f \in L_1(\mathbf{T})$. Note that \tilde{f} is not necessarily integrable whenever f is.

We will prove also that the operators U_n^θ and \tilde{U}_n^θ ($n \in \mathbf{N}$) are uniformly bounded in n from $H_{p,q}(\mathbf{T})$ to $H_{p,q}(\mathbf{T})$ ($p_0 < p < \infty$, $0 < q \leq \infty$) and are uniformly of weak type $(H_{p_0}(\mathbf{T}), H_{p_0}(\mathbf{T}))$. From this it follows that $U_n^\theta f \rightarrow f$ and $\tilde{U}_n^\theta f \rightarrow \tilde{f}$ in $H_{p,q}(\mathbf{T})$ norm (resp. in weak $H_{p_0}(\mathbf{T})$ norm) as $n \rightarrow \infty$, whenever $f \in H_{p,q}(\mathbf{T})$ ($p_0 < p < \infty$, $0 < q \leq \infty$) (resp. $f \in H_{p_0}(\mathbf{T})$).

As special case we investigate ten well known summability methods, amongst others the summations mentioned above.

We consider also the θ -means of Fourier transforms on the real line and prove all the results above in this context.

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2. HARDY SPACES AND CONJUGATE FUNCTIONS

Let \mathbf{N} denote the none-negative integers, \mathbf{R} the real numbers; \mathbf{R}_+ the positive real numbers, $\mathbf{T} := [-\pi, \pi)$ and λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I . We briefly write $L_{p,q}(\mathbf{X})$ instead of the real Lorentz space $L_{p,q}(\mathbf{X}, \lambda)$ ($0 < p, q \leq \infty$) and its norm is denoted by $\|\cdot\|_{p,q}$ where $\mathbf{X} = \mathbf{T}$ or \mathbf{R} (for the exact definitions see e.g. Weisz [21] and the references there). We extend all functions on \mathbf{T} periodically to \mathbf{R} .

Let f be a distribution on $C^\infty(\mathbf{T})$. The n th Fourier coefficient is defined by $\hat{f}(n) := f(e^{-inx})$ where $i = \sqrt{-1}$. In special case, if f is an integrable function then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbf{T}} f(x) e^{-inx} dx \quad (n \in \mathbf{N}).$$

The non-tangential maximal function of a distribution f is defined by

$$f^*(x) := \sup_{0 < r < 1} |f * P_r(x)|,$$

where $*$ denotes the convolution and

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{ikx} = \frac{1-r^2}{1+r^2-2r \cos x} \quad (x \in \mathbf{T})$$

is the Poisson kernel.

For $0 < p, q \leq \infty$ the *Hardy-Lorentz space* $H_{p,q}(\mathbf{T})$ consists of all distributions f for which

$$\|f\|_{H_{p,q}(\mathbf{T})} := \|f^*\|_{p,q} < \infty.$$

Note that in case $p = q$ the usual definition of Hardy spaces $H_p(\mathbf{T}) = H_{p,p}(\mathbf{T})$ is obtained. For other equivalent definitions we call for Fefferman and Stein [5] and Stein [15]. Recall that $L_1(\mathbf{T}) \subset H_{1,\infty}(\mathbf{T})$, more exactly,

$$\|f\|_{H_{1,\infty}(\mathbf{T})} = \sup_{\rho > 0} \rho \lambda(f^* > \rho) \leq \|f\|_1 \quad (f \in L_1(\mathbf{T})). \quad (1)$$

Moreover,

$$H_{p,q}(\mathbf{T}) \sim L_{p,q}(\mathbf{T}) \quad (1 < p < \infty, 0 < q \leq \infty), \quad (2)$$

where \sim denotes the equivalence of the norms and spaces (see Fefferman and Stein [5], Stein [15], Fefferman, Riviere, Sagher [4]).

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Fefferman, Riviere, Sagher [4] and also Weisz [19]).

THEOREM A. *If a sublinear (resp. linear) operator V is bounded from $H_{p_0}(\mathbf{T})$ to $L_{p_0}(\mathbf{T})$ (resp. to $H_{p_0}(\mathbf{T})$) and from $L_{p_1}(\mathbf{T})$ to $L_{p_1}(\mathbf{T})$ ($p_0 \leq 1 < p_1 \leq \infty$) then it is also bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ (resp. to $H_{p,q}(\mathbf{T})$) if $p_0 < p < p_1$ and $0 < q \leq \infty$.*

For a distribution

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

the conjugate distribution is defined by

$$\tilde{f} \sim \sum_{k=-\infty}^{\infty} (-i \operatorname{sign} k) \hat{f}(k) e^{ikx}.$$

As is well known, if f is an integrable function then

$$\tilde{f}(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(x-t)}{2 \tan(t/2)} dt := \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{\varepsilon < |t| < \pi} \frac{f(x-t)}{2 \tan(t/2)} dt.$$

Moreover, the conjugate function \tilde{f} does exist almost everywhere, but it is not integrable in general. It is easy to see that $(\tilde{f})^\sim = -f$.

Fefferman and Stein [5] verified that

$$\|f\|_{H_p(\mathbf{T})} \sim \|f\|_p + \|\tilde{f}\|_p \quad (0 < p < \infty). \quad (3)$$

3. θ -SUMMABILITY OF FOURIER SERIES

First we introduce the Fourier transform for an integrable function $f \in L_1(\mathbf{R})$ by

$$\hat{f}(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-iux} dx \quad (u \in \mathbf{R}).$$

The θ -summation was considered in Butzer and Nessel [3] and, more recently Bokor, Schipp, Szili and Vértesi [2, 11, 12, 16, 17] investigated the uniform convergence of the θ -means and some interpolation problems for continuous functions.

In what follows we suppose that $\theta \in L_1(\mathbf{R})$ is an even continuous function satisfying $\theta(0) = 1$, $\hat{\theta} \in L_1(\mathbf{R})$ and $\theta(\frac{\cdot}{n+1}) \in l_1$. Note that this last condition is satisfied if θ is non-increasing on \mathbf{R}_+ or if it has compact support.

Denote by $s_n f$ the n th partial sum of the Fourier series of a distribution f , namely,

$$s_n f(x) := \sum_{k=-n}^n \hat{f}(k) e^{ikx}.$$

The θ -means of a distribution f are defined by

$$U_n^\theta f(x) := \sum_{k=-\infty}^{\infty} \theta\left(\frac{k}{n+1}\right) \hat{f}(k) e^{ikx} = (f * K_n^\theta)(x) \quad (x \in \mathbf{T}), \quad (4)$$

where the K_n^θ kernels satisfy

$$K_n^\theta(t) := \sum_{k=-\infty}^{\infty} \theta\left(\frac{k}{n+1}\right) e^{ikt} = 1 + 2 \sum_{k=1}^{\infty} \theta\left(\frac{k}{n+1}\right) \cos(kt) \quad (t \in \mathbf{T}).$$

It is easy to see that if θ has bounded variation then the θ -summation is permanent, i.e. if $s_n f$ is convergent in some sense then $U_n^\theta f$ is also convergent and converges to the same limit.

Following Butzer and Nessel [3] and Schipp and Bokor [11] we verify a new characterization for the θ -means. We write $U_n^\theta f$ as a singular integral of f and the Fourier transform of θ over the whole real line.

LEMMA 1. *If $f \in L_1(\mathbf{T})$ then*

$$U_n^\theta f(x) = (n+1) \int_{-\infty}^{\infty} f(x-t) \hat{\theta}((n+1)t) dt \quad (n \in \mathbf{N}). \quad (5)$$

Proof. If $f(t) = e^{ikt}$ then

$$\begin{aligned} (n+1) \int_{-\infty}^{\infty} e^{ik(x-t)} \hat{\theta}((n+1)t) dt &= e^{ikx} \int_{-\infty}^{\infty} e^{-ikt/(n+1)} \hat{\theta}(t) dt \\ &= \theta\left(\frac{k}{n+1}\right) e^{ikx} = U_n^\theta f(x). \end{aligned}$$

Hence the lemma holds also for trigonometric polynomials. Let f be an arbitrary element from $L_1(\mathbf{T})$ and (f_k) be a sequence of trigonometric

polynomials such that $f_k \rightarrow f$ in $L_1(\mathbf{T})$ norm. The condition $\theta(\frac{\cdot}{n+1}) \in l_1$ implies that $K_n^\theta \in L_1(\mathbf{T})$. Since

$$U_n^\theta f(x) = \frac{1}{2\pi} \int_{\mathbf{T}} f(x-t) K_n^\theta(t) dt$$

for $f \in L_1(\mathbf{T})$, we can conclude that $U_n^\theta f_k \rightarrow U_n^\theta f$ in $L_1(\mathbf{T})$ norm as $k \rightarrow \infty$. On the other hand, $\hat{\theta} \in L_1(\mathbf{R})$, and so

$$(n+1) \int_{-\infty}^{\infty} f_k(x-t) \hat{\theta}((n+1)t) dt \rightarrow (n+1) \int_{-\infty}^{\infty} f(x-t) \hat{\theta}((n+1)t) dt$$

in $L_1(\mathbf{T})$ norm as $k \rightarrow \infty$. This finishes the proof of the lemma. ■

The *conjugate θ -means* of a distribution f are introduced by

$$\tilde{U}_n^\theta f(x) := \sum_{k=-\infty}^{\infty} \theta\left(\frac{k}{n+1}\right) (-i \operatorname{sign} k) \hat{f}(k) e^{ikx}.$$

The *maximal* and *conjugate maximal θ -operators* are defined by

$$U_*^\theta f := \sup_{n \in \mathbf{N}} |U_n^\theta f| \quad \text{and} \quad \tilde{U}_*^\theta f := \sup_{n \in \mathbf{N}} |\tilde{U}_n^\theta f|,$$

respectively. Our first boundedness result is the following

LEMMA 2. *The operator U_*^θ is bounded on $L_\infty(\mathbf{T})$.*

Proof. The characterization (5) implies that

$$\|U_n^\theta f\|_\infty \leq \|f\|_\infty \|\hat{\theta}\|_1$$

for all $n \in \mathbf{N}$, which shows the lemma. ■

In this paper the constants C are depending only on θ and the constants C_p (resp. $C_{p,q}$) are depending only on p and θ (resp. p, q and θ) and may denote different constants in different contexts.

4. THE BOUNDEDNESS OF THE MAXIMAL θ -OPERATOR

A *generalized interval* on \mathbf{T} is either an interval $I \subset \mathbf{T}$ or $I = [-\pi, x) \cup [y, \pi)$. A bounded measurable function a is a *p-atom* if there exists a generalized interval I such that

- (i) $\int_I a(x) x^j dx = 0$ where $j \in \mathbf{N}$ and $j \leq [1/p - 1]$, the integer part of $(1/p - 1)$,
- (ii) $\|a\|_\infty \leq |I|^{-1/p}$,
- (iii) $\{a \neq 0\} \subset I$.

We will use the following two theorems, the first one can be found in Weisz [21].

THEOREM B. *Suppose that the operator V is sublinear and, for some $0 < p \leq 1$,*

$$\int_{\mathbf{T} \setminus 8I} |Va|^p d\lambda \leq C_p \quad (6)$$

for every p -atom a where I is the support of the atom and $8I$ is the generalized interval with the same center as I and with length $8|I|$. If V is bounded from $L_{p_1}(\mathbf{T})$ to $L_{p_1}(\mathbf{T})$ for a fixed $1 < p_1 \leq \infty$ then

$$\|Vf\|_p \leq C_p \|f\|_{H_p(\mathbf{T})} \quad (f \in H_p(\mathbf{T})).$$

We formulate also a weak version of this theorem, which is an easy modification of a result in Long [8], so we sketch the proof, only.

THEOREM C. *Suppose that the operator V is sublinear and, for some $0 < p < 1$,*

$$\sup_{\rho > 0} \rho^p \lambda(\{|Va| > \rho\} \cap \{\mathbf{T} \setminus 8I\}) \leq C_p \quad (7)$$

for every p -atom a where I denotes again the support of the atom. If V is bounded from $L_{p_1}(\mathbf{T})$ to $L_{p_1}(\mathbf{T})$ for a fixed $1 < p_1 \leq \infty$ then

$$\|Vf\|_{p, \infty} \leq C_p \|f\|_{H_p(\mathbf{T})} \quad (f \in H_p(\mathbf{T})).$$

Proof. If (7) is satisfied without the intersection with $\{\mathbf{T} \setminus 8I\}$, then the result can be found in Long [8, p. 279]. Then

$$\begin{aligned} & \sup_{\rho > 0} \rho^p (\{|Va| > \rho\} \cap \{8I\}) \\ & \leq \int_{8I} |Va|^p d\lambda \leq C_p \left(\int_{\mathbf{T}} |Va|^{p_1} d\lambda \right)^{p/p_1} |I|^{1-p/p_1} \leq C_p, \end{aligned}$$

which proves the theorem. \blacksquare

Now we are ready to prove the boundedness of the maximal operator on the Hardy spaces. First we recall a known result, which was shown in another context. Taking into account (5) we can see that Torchinsky [18, p. 82–84] has proved essentially the next inequality.

PROPOSITION 1. *Assume that there is an even, on \mathbf{R}_+ non-increasing function η_0 such that $|\hat{\theta}| \leq \eta_0$. If $\eta_0 \in L_1(\mathbf{R})$ then*

$$\sup_{\rho > 0} \rho \lambda(U_*^\theta f > \rho) \leq C \|f\|_1 \quad (f \in L_1(\mathbf{T})). \quad (8)$$

It follows from Proposition 1 and Lemma 2 and by interpolation that

$$\|U_*^\theta f\|_p \leq C \|f\|_p \quad (f \in L_p(\mathbf{T}), 1 < p \leq \infty).$$

If we suppose a little bit more on η_0 then we can prove that U_*^θ is bounded from $H_1(\mathbf{T})$ to $L_1(\mathbf{T})$.

THEOREM 1. *Assume that there is an even, on \mathbf{R}_+ non-increasing function η_0 such that $|\hat{\theta}| \leq \eta_0$, $t\eta_0(t)$ is non-increasing on the interval $[1, \infty)$. If θ has compact support that is contained in $[-c, c]$ and if $\eta_0 \in L_1(\mathbf{R})$ then*

$$\|_*^\theta f\|_1 \leq cC \|f\|_{H_1(\mathbf{T})} \quad (f \in H_1(\mathbf{T})).$$

Proof. We will verify (6) for $p=1$. Then Theorem 1 will follow from Lemma 2 and Theorem B.

To this end let a be an arbitrary 1-atom with support I and $2^{-K-1} < |I|/\pi \leq 2^{-K}$ ($K \in \mathbf{N}$). If t_0 is the center of I , then the center of $I' := I - t_0$ is zero. By changing the variables we can see that

$$\begin{aligned} \int_{\mathbf{T} \setminus 8I} |U_*^\theta a|^p d\lambda &= \int_{\mathbf{T} \setminus 8I} \sup_{n \in \mathbf{N}} \left| \int_I a(t) K_n^\theta(x-t) dt \right|^p dx \\ &= \int_{\mathbf{T} \setminus 8I'} \sup_{n \in \mathbf{N}} \left| \int_{I'} a'(t) K_n^\theta(x-t) dt \right|^p dx \\ &= \int_{\mathbf{T} \setminus 8I'} |U_*^\theta a'|^p d\lambda, \end{aligned}$$

where $a'(t) := a(t + t_0)$.

Hence we can suppose that the center of I is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

First suppose that $x \geq 0$. Then

$$\begin{aligned} \int_{\{\mathbf{T} \setminus 8I\} \cap \{x \geq 0\}} |U_n^\theta a(x)| dx &\leq \sum_{i=2}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1) 2^{-K}} \sup_{n+1 \geq r_i} |U_n^\theta a(x)| dx \\ &\quad + \sum_{i=2}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1) 2^{-K}} \sup_{n+1 < r_i} |U_n^\theta a(x)| dx \\ &= (A) + (B), \end{aligned}$$

where $r_i := 2^K i^{-\alpha}$ ($i \in \mathbf{N}$) with $\alpha > 0$ chosen later.

By Lemma 1 and by the condition $|\hat{\theta}| \leq \eta_0$ we estimate $U_n^\theta a$ as follows:

$$|U_n^\theta a(x)| \leq (n+1) \int_{-\infty}^{\infty} |a(t)| \eta_0((n+1)(x-t)) dt.$$

Then, by the definition of the 1-atom,

$$\sup_{n+1 \geq r_i} |U_n^\theta a(x)| \leq 2^K r_i \sum_{k=-\infty}^{\infty} \int_{I+2k\pi}^{\infty} \eta_0(r_i(x-t)) dt := (A_1)(x) + (A_2)(x),$$

where (A_1) denotes the term $k=0$ and (A_2) the sum $\sum_{|k|=1}^{\infty}$.

If $t \in I$ and $x \in [\pi i 2^{-K}, \pi(i+1) 2^{-K}]$ for some $i=2, \dots, 2^K-1$, then

$$|x-t| \geq \pi i 2^{-K} - \pi 2^{-K-1} \geq \pi(i-1) 2^{-K}. \quad (9)$$

This implies

$$(A_1)(x) \leq C 2^K i^{-\alpha} \eta_0(i^{1-\alpha} \pi/2) \quad (x \in [\pi i 2^{-K}, \pi(i+1) 2^{-K}]).$$

If $t \in I + 2k\pi$ for some $k \neq 0$ then $|x-t| \sim 2|k|\pi$. We have

$$(A_2)(x) \leq C 2^K i^{-\alpha} \sum_{k=1}^{\infty} \eta_0(2^{K+1} i^{-\alpha} \pi k) \leq C \int_0^{\infty} \eta_0 d\lambda \leq C.$$

Hence, in case $0 < \alpha < 1$,

$$(A)(x) \leq C + C 2^{-K} \sum_{i=2}^{2^K-1} 2^K i^{-\alpha} \eta_0(i^{1-\alpha} \pi/2) \leq C \int_0^{\infty} \eta_0 d\lambda \leq C.$$

Now let us consider (B) . Since $\text{supp } \theta \subset [-c, c]$ and θ is bounded, (4) implies

$$|U_n a(x)| \leq C \sum_{|k|=0}^{c(n+1)} |\hat{a}(k)|.$$

As

$$|\hat{a}(k)| = \left| \frac{1}{2\pi} \int_I a(x)(e^{-ikx} - 1) dx \right| \leq C \int_I |a(x)| |kx| dx \leq C |k| |I|$$

we obtain

$$\sup_{n+1 < r_i} |U_n a(x)| \leq c C r_i^2 2^{-K} \leq c C 2^K i^{-2\alpha}.$$

Therefore,

$$(B) \leq c C 2^{-K} \sum_{i=2}^{2^K-1} 2^{Ki-2\alpha}$$

which is bounded if $1/2 < \alpha < 1$. If $x < 0$ then

$$\begin{aligned} \int_{\{\mathbf{T} \setminus 8I\} \cap \{x < 0\}} |U_*^\theta a(x)| dx &\leq \sum_{i=-2}^{-(2^K-1)} \int_{\pi i 2^{-K}}^{\pi(i-1) 2^{-K}} \sup_{n+1 \geq r_i} |U_n^\theta a(x)| dx \\ &\quad + \sum_{i=-2}^{-(2^K-1)} \int_{\pi i 2^{-K}}^{\pi(i-1) 2^{-K}} \sup_{n+1 < r_i} |U_n^\theta a(x)| dx, \end{aligned}$$

where $r_i := 2^K |i|^{-\alpha}$. The inequality

$$\int_{\{\mathbf{T} \setminus 8I\} \cap \{x < 0\}} |U_*^\theta a(x)| dx \leq c C$$

can be proved exactly as above. The proof of the theorem is complete. \blacksquare

Remark. We can extend this result to $p < 1$ as follows. In addition to the conditions of Theorem 1 suppose that

$$\int_0^\infty t^{(1-p_0)(1+\varepsilon)/(2p_0-1)} \eta_0(t)^{p_0} dt < \infty \quad (10)$$

for some $1/2 < p_0 < 1$ and $\varepsilon > 0$. Then we can prove in the same way that

$$\|U_*^\theta f\|_{p_0} \leq C_{p_0} \|f\|_{H_{p_0}(\mathbf{T})} \quad (f \in H_{p_0}(\mathbf{T}))$$

and

$$\|U_*^\theta f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbf{T})} \quad (f \in H_{p,q}(\mathbf{T}))$$

for every $p_0 < p < \infty$ and $0 < q \leq \infty$.

If $p_0 = 1$ then condition (10) reduces to the integrability of η_0 . Since $\hat{\theta}$ is bounded, we may suppose that η_0 is also bounded. It is easy to see that if (10) is satisfied for p_0 then it holds also for all $p_0 \leq p \leq 1$. The interval $[1, \infty)$ in Theorem 1 is a technical condition, only, we could change it to $[c, \infty)$ for any $c > 0$.

If we have some information about the derivatives of $\hat{\theta}$ we can prove an even sharper result. Let $\hat{\theta}^{(k)}$ be denote the k th derivative of $\hat{\theta}$.

THEOREM 2. *Assume that there are two even, on \mathbf{R}_+ non-increasing functions η_N and η_{N+1} such that $|\hat{\theta}^{(N)}| \leq \eta_N$, $0 \neq |\hat{\theta}^{(N+1)}| \leq \eta_{N+1}$, $t^{N+1}\eta_N(t)$ is non-increasing on $[1, \infty)$ and $t^{N+2}\eta_{N+1}(t)$ is non-decreasing on \mathbf{R}_+ ($N \in \mathbf{N}$). If $\eta_N, \eta_{N+1} \in L_{p_0}(\mathbf{R})$ for some $p_0 \leq 1/(N+1)$ then*

$$\|U_*^\theta f\|_{p_0} \leq C_{p_0} \|f\|_{H_{p_0}(\mathbf{T})} \quad (f \in H_{p_0}(\mathbf{T})) \quad (11)$$

and

$$\|U_*^\theta f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbf{T})} \quad (f \in H_{p,q}(\mathbf{T})) \quad (12)$$

for every $p_0 < p < \infty$, $0 < q \leq \infty$. In particular, if $f \in L_1(\mathbf{T})$ then (8) holds. Moreover, if $\eta_N, \eta_{N+1} \in L_{p_0, \infty}(\mathbf{R})$ for some $p_0 \leq 1/(N+1)$ ($p_0 \neq 1$) then

$$\|U_*^\theta f\|_{p_0, \infty} \leq C_{p_0} \|f\|_{H_{p_0}(\mathbf{T})} \quad (f \in H_{p_0}(\mathbf{T})) \quad (13)$$

and (12) and (8) are valid.

Proof. First we show (6) for $p = p_0$. Let a be an arbitrary p_0 -atom with support I and $2^{-K-1} < |I|/\pi \leq 2^{-K}$ ($K \in \mathbf{N}$). As in the proof of Theorem 1 we can suppose that the center of I is zero. Let $A_0(x) := a(x)$ ($x \in \mathbf{R}$) and

$$A_j(x) := \int_{-\infty}^x A_{j-1}(t) dt \quad (x \in \mathbf{R}; j = 1, \dots, [1/p - 1] + 1).$$

By (i) of the definition of the atom we can show that $\text{supp } A_j \subset \bigcup_{k=-\infty}^{\infty} \{I + 2k\pi\}$ ($j = 1, \dots, [1/p - 1] + 1$). Moreover, by (ii),

$$\|A_j\|_{\infty} \leq |I|^{-1/p+j} \quad (j = 1, \dots, [1/p - 1] + 1). \quad (14)$$

Using Lemma 1 and integrating by parts we can see that

$$\begin{aligned} |U_n^\theta a(x)| &= (n+1)^{N+1} \left| \int_{-\infty}^{\infty} A_N(t) \hat{\theta}^{(N)}((n+1)(x-t)) dt \right| \\ &\leq (n+1)^{N+1} \int_{-\infty}^{\infty} |A_N(t)| \eta_N((n+1)(x-t)) dt. \end{aligned} \quad (15)$$

By the conditions of the theorem and (14),

$$\begin{aligned} \sup_{n+1 \geq 2^K} |U_n^\theta a(x)| &\leq 2^{K/p_0+K} \sum_{k=-\infty}^{\infty} \int_{I+2k\pi}^{\infty} \eta_N(2^K(x-t)) dt \\ &= (C)(x) + (D)(x), \end{aligned}$$

where (C) denotes the term $k=0$ and (B) the sum $\sum_{|k|=1}^{\infty}$.

We suppose again that $x \in [-\pi, \pi) \setminus 8I$ and $x \geq 0$. If $t \in I$ and $x \in [\pi i 2^{-K}, \pi(i+1) 2^{-K})$ ($i=2, \dots, 2^K-1$), then (9) implies

$$(C)(x) \leq C 2^{K/p_0} \eta_{N+1}((i-1)\pi) \quad (x \in [\pi i 2^{-K}, \pi(i+1) 2^{-K})),$$

thus

$$\begin{aligned} \int_{\{\mathbf{T} \setminus 8I\} \cap \{x \geq 0\}} (C)(x)^{p_0} dx &\leq C_{p_0} 2^{-K} \sum_{i=2}^{2^K-1} 2^K \eta_{N+1}^{p_0}((i-1)\pi) \\ &\leq C_{p_0} \int_0^\infty \eta_{N+1}^{p_0} d\lambda \leq C_{p_0}. \end{aligned}$$

If $t \in I + 2k\pi$ ($k \neq 0$) then

$$(D)^{p_0}(x) \leq C_{p_0} 2^K \sum_{k=1}^{\infty} \eta_{N+1}^{p_0}(2^{K+1}\pi k) \leq C_{p_0} \int_\varepsilon^\infty \eta_{N+1}^{p_0} d\lambda \leq C_{p_0}$$

and (6) is satisfied.

Similarly to (15) we can also obtain that

$$|U_n^\theta a(x)| \leq (n+1)^{N+2} \int_{-\infty}^{\infty} |A_{N+1}(t)| \eta_{N+1}((n+1)(x-t)) dt$$

and then $\sup_{n+1 < 2^K} |U_n^\theta a(x)|$ can be estimated in the same way as $\sup_{n+1 \geq 2^K} |U_n^\theta a(x)|$ above. The case $x < 0$ can be treated similarly. This proves inequality (11).

To prove (13) observe that

$$\begin{aligned} \rho^{p_0} \lambda(\{(C) > \rho\} \cap \{\mathbf{T} \setminus 8I\}) &= \rho^{p_0} \sum_{i \geq 1: \eta_{N+1}(i\pi) > \rho 2^{-K/p_0}} 2^{-K} \\ &\leq C_\varepsilon \rho^{p_0} 2^{-K} \lambda(\{\eta_{N+1} > \rho 2^{-K/p_0}\}) \\ &\leq C_\varepsilon \|\eta_{N+1}\|_{L_{p_0, \infty}(\mathbf{R})}^{p_0}. \end{aligned}$$

Obviously, (D) satisfies also (7). We can estimate $\sup_{n+1 < 2^K} |U_n^\theta a(x)|$ similarly, which shows (13). The inequality (12) follows from Theorem A. Applying (1) and (12) for $p=1$ and $q=\infty$, we conclude

$$\|U_*^\theta f\|_{1, \infty} \leq C \|f\|_{H_{1, \infty}} \leq C \|f\|_1$$

which shows (8). This finishes the proof of the theorem. ■

Remark. We can weaken the condition $t^{N+1}\eta_N(t) \searrow$ in Theorem 2 by

$$t^{N+1}\eta_N(t) \leq t_0^{N+1}\eta_N(t_0) \quad (t \geq t_0 \geq 1).$$

Of course we could replace $t^{N+2}\eta_{N+1}(t) \nearrow$ also by an analogous condition.

In the next theorem we show the boundedness of U_*^θ on Hardy spaces if $t^{N+2}\hat{\theta}^{(N+1)}(t)$ is bounded.

THEOREM 3. Assume that $0 \neq |t^{N+2}\hat{\theta}^{(N+1)}(t)| \leq C$ for some $N \in \mathbf{N}$. Then (12) holds for every $1/(N+2) < p < \infty$, $0 < q \leq \infty$ and

$$\|U_*^\theta f\|_{1/(N+2), \infty} \leq C_{1/(N+2)} \|f\|_{H_{1/(N+2)}(\mathbf{T})} \quad (f \in H_{1/(N+2)}(\mathbf{T})). \quad (16)$$

Especially, if $f \in L_1(\mathbf{T})$ then the weak type $(1, 1)$ inequality (8) holds.

Proof. First we show (12) for $1/(N+2) < p = q \leq 1/(N+1)$. The general case of (12) will follow from Theorem A. To this end, by Lemma 1 and Theorem B we have only to prove that condition (6) is satisfied for $1/(N+2) < p \leq 1/(N+1)$. Note that in this case $[1/p - 1] = N$.

Let a be an arbitrary p -atom with support I and $2^{-K-1} < |I|/\pi \leq 2^{-K}$ ($K \in \mathbf{N}$). We can suppose again that the center of I is zero. As in (15) we conclude

$$\begin{aligned} |U_n^\theta a(x)| &= (n+1)^{N+2} \left| \int_{-\infty}^{\infty} A_{N+1}(t) \hat{\theta}^{(N+1)}((n+1)(x-t)) dt \right| \\ &\leq |I|^{-1/p+N+1} \sum_{k=-\infty}^{\infty} \int_{I+2k\pi} |x-t|^{-(N+2)} dt \\ &:= (E)(x) + (F)(x), \end{aligned}$$

where (E) denotes again the term corresponding to $k=0$ and (F) the sum $\sum_{|k|=1}^{\infty}$.

If $t \in I$ and $x \in [\pi i 2^{-K}, \pi(i+1) 2^{-K})$ for some $i=2, \dots, 2^K-1$, then (9) implies

$$(E)(x) \leq C 2^{K/p} i^{-(N+2)} \quad (x \in [\pi i 2^{-K}, \pi(i+1) 2^{-K})) \quad (17)$$

and so

$$\begin{aligned} \int_{\{\mathbf{T} \setminus 8I\} \cap \{x \geq 0\}} (E)(x)^p dx &\leq \sum_{i=2}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1) 2^{-K}} (E)(x)^p dx \\ &\leq C_p 2^{-K} \sum_{i=2}^{2^K-1} 2^{Ki-(N+2)p} \leq C_p. \end{aligned}$$

If $t \in I + 2k\pi$ for some $k \neq 0$ then

$$(F)(x) \leq C 2^{-K(-1/p+N+2)} \sum_{k=1}^{\infty} k^{-(N+2)} \leq C$$

and (6) is satisfied automatically. If $x < 0$ then we can show (6) in the same way.

To prove (16) we have to check (7) for $p = 1/(N+2)$. Inequality (17) implies that

$$\rho^{1/(N+2)} \lambda(\{(E) > \rho\} \cap \{\mathbf{T} \setminus 8I\}) = \rho^{1/(N+2)} \sum_{i=1}^{(2^{K(N+2)} \rho^{-1})^{1/(N+2)}} 2^{-K} \leq 1.$$

Since (F) satisfies also (7), we have shown (16). Inequality (8) can be verified by interpolation as in Theorem 2. The proof of the theorem is complete. ■

Notice that by interpolation we get the inequality

$$\|f\|_{H_{p,q}(\mathbf{T})} \sim \|f\|_{p,q} + \|\tilde{f}\|_{p,q} \quad (0 < p < \infty, 0 < q \leq \infty)$$

from (3). Since $\|f\|_{H_{p,q}} \sim \|\tilde{f}\|_{H_{p,q}}$ and $\tilde{U}_n^\theta f = U_n^\theta \tilde{f}$, we can extend Theorems 2 and 3 easily to the conjugate maximal operators and to the θ -means as follows.

THEOREM 4. *Theorems 2 and 3 hold also for the operator \tilde{U}_*^θ instead of U_*^θ . Moreover, if we replace on the left hand side of the inequalities (11)–(13) and (16) the operator U_*^θ by U_n^θ or \tilde{U}_n^θ and the space $L_{p,q}$ by $H_{p,q}$, then these inequalities hold uniformly in n .*

Since the trigonometric polynomials are dense in $L_1(\mathbf{T})$ and in the Hardy spaces, the inequalities of Theorems 2-4 and the usual density argument (see Marcinkiewicz, Zygmund [9]) imply

COROLLARY 1. *Under the conditions of Theorem 2 or 3, $f \in L_1(\mathbf{T})$ implies*

$$U_n^\theta f \rightarrow f \text{ a.e.} \quad \text{and} \quad \tilde{U}_n^\theta f \rightarrow \tilde{f} \text{ a.e.} \quad \text{as } n \rightarrow \infty.$$

Moreover, if e.g. (12) is satisfied, then these two convergence hold in $H_{p,q}$ norm, whenever $f \in H_{p,q}$, if (13) is true, then in $H_{p_0, \infty}$ norm, whenever $f \in H_{p_0}$. From Theorems 3 and 4 we obtain similar convergence results, i.e. if (16) is satisfied.

Note that \tilde{f} is not necessarily integrable whenever f is.

5. APPLICATIONS TO VARIOUS SUMMABILITY METHODS

In this section we consider several summability methods introduced in the book of Butzer and Nessel [3] and some other popular ones as special cases of the θ -summation. Of course, there are a lot of other summability methods which could be considered as special cases. The elementary computations in the examples below are left to the reader.

Let C_0 consists of all continuous functions f , for which $\lim_{|x| \rightarrow \infty} f(x) = 0$. Butzer and Nessel [3] verified that if $\theta \in C_0$ and θ , θ' and $x\theta''(x)$ are integrable functions, then $\hat{\theta} \in L_1(\mathbf{R})$.

LEMMA 3. Suppose that $\theta \in L_1(\mathbf{R}) \cap C_0$ is even and each term of $(x^i \theta(x))^{(i+1)}$ is integrable for some $i \geq 0$. Then $\hat{\theta} \in L_1(\mathbf{R})$ and

$$|\hat{\theta}^{(i)}(x)| \leq \frac{C}{x^{i+1}} \quad (x \in (0, \infty)).$$

Proof. The integrability of $\hat{\theta}$ comes from the result above. By integrating by parts we have

$$\begin{aligned} |\hat{\theta}^{(i)}(x)| &= \left| \int_0^\infty t^i \theta(t) \cos tx \, dt \right| = \frac{1}{x} \left| \int_0^\infty (t^i \theta(t))' \sin tx \, dt \right| = \dots \\ &= \frac{1}{x^{i+1}} \left| [\theta(t) \cos tx]_0^\infty \right| + \frac{1}{x^{i+1}} \left| \int_0^\infty (t^i \theta(t))^{(i+1)} \cos tx \, dt \right|. \end{aligned}$$

Of course, in the last line probably cos have to be changed to sin. ■

Our first three examples satisfy the conditions of Lemma 3.

EXAMPLE 1. *Weierstrass summation.* Let $\theta_1(x) = e^{-|x|^\gamma}$ for some $0 < \gamma < \infty$. It is easy to see that $(x^i e^{-|x|^\gamma})^{(i+1)} \in L_1(\mathbf{R})$ for all $i \geq 0$. The θ -means are given by

$$U_n^{\theta_1} f(x) := \sum_{k=-\infty}^{\infty} e^{-(|k|/(n+1))^\gamma} \hat{f}(k) e^{ikx}.$$

Of course, we can take another index set than \mathbf{N} . For example we can change $(\frac{1}{n+1})^\gamma$ to t :

$$V_t^{\theta_1} f(x) := \sum_{k=-\infty}^{\infty} e^{-t|k|^\gamma} \hat{f}(k) e^{ikx} \quad (t \in (0, \infty)),$$

or e^{-t} by r :

$$W_r^{\theta_1} f(x) := \sum_{k=-\infty}^{\infty} r^{|k|^\gamma} \hat{f}(k) e^{ikx} \quad (r \in (0, 1)).$$

By Lemma 3, θ_1 satisfies the conditions of Theorem 3 for all $N \in \mathbf{N}$. This means e.g. that the operators $U_*^{\theta_1}$, $V_*^{\theta_1}$ and $W_*^{\theta_1}$ are bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ for every $0 < p < \infty$ and $0 < q \leq \infty$. Moreover, $U_n^{\theta_1} f \rightarrow f$ a.e. as $n \rightarrow \infty$, $V_t^{\theta_1} f \rightarrow f$ a.e. as $t \rightarrow 0$ and $W_r^{\theta_1} f \rightarrow f$ a.e. as $r \rightarrow 1$. If $\gamma = 1$ then this last result is the well known convergence of the Abel means.

EXAMPLE 2. *Picar summation.* Let $\theta_2(x) = (1 + |x|^\gamma)^{-1}$ for some $1 < \gamma < \infty$. One can check that $(x^i(1 + |x|^\gamma)^{-1})^{(i+1)} \in L_1(\mathbf{R})$ for all $i \geq 0$. The θ -means are given by

$$U_n^{\theta_2} f(x) := \sum_{k=-\infty}^{\infty} \frac{1}{1 + \left(\frac{|k|}{n+1}\right)^\theta} \hat{f}(k) e^{ikx}.$$

It follows from Lemma 3 that Theorems 3 and 4 and Corollary 1 hold for this summability method. For example, $U_*^{\theta_2}$ is bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ for every $0 < p < \infty$ and $0 < q \leq \infty$.

EXAMPLE 3. *Bessel assumption.* Let $\theta_3(x) = (1 + x^2)^{-\gamma/2}$ for some $1 < \gamma < \infty$. Again, $(x^i(1 + x^2)^{-\gamma/2})^{(i+1)} \in L_1(\mathbf{R})$ for all $i \geq 0$. The θ -means are given by

$$U_n^{\theta_3} f(x) := \sum_{k=-\infty}^{\infty} \frac{1}{\left(1 + \left(\frac{k}{n+1}\right)^2\right)^{\gamma/2}} \hat{f}(k) e^{ikx}.$$

Thus Theorems 3 and 4 and Corollary 1 hold again for every $0 < p < \infty$ and $0 < q \leq \infty$.

The next six θ -functions are special cases of Theorem 3.

EXAMPLE 4. Fejér summation. Let

$$\theta_4(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

$U_n^{\theta_4}$ is the n th Fejér operator:

$$U_n^{\theta_4} f(x) := \sum_{k=-n}^n \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k) e^{ikx} = \frac{1}{n+1} \sum_{k=0}^n s_k f(x).$$

It is known that

$$\hat{\theta}_4(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin x/2}{x/2} \right)^2$$

and $|\hat{\theta}'_4(x)| \leq C/x^2$. Consequently, Theorems 3 and 4 and Corollary 1 hold for $N=0$.

EXAMPLE 5. Riemann summation. Let

$$\theta_5(x) = \left(\frac{\sin x/2}{x/2} \right)^2 = \sqrt{2\pi} \hat{\theta}_4(x).$$

Then $\hat{\theta}_5(x) = \sqrt{2\pi} \theta_4(x) = \sqrt{2\pi} \max(0, 1 - |x|)$ and so $|\hat{\theta}'_5(x)| = 1_{(-1,1)}(x) \leq C/x^2$. The Riemann means are given by

$$U_n^{\theta_5} f(x) := \sum_{k=-\infty}^{\infty} \left(\frac{\sin k/(2(n+1))}{k/(2(n+1))} \right)^2 \hat{f}(k) e^{ikx}.$$

If we change $1/(n+1)$ to μ then

$$V_{\mu}^{\theta_5} f(x) := \sum_{k=-\infty}^{\infty} \left(\frac{\sin k\mu/2}{k\mu/2} \right)^2 \hat{f}(k) e^{ikx} \quad (\mu \in (0, \infty)).$$

This yields that the operators $U_*^{\theta_5}$ and $V_*^{\theta_5}$ are bounded from $H_{p,q}(\mathbf{T})$ to $L_{p,q}(\mathbf{T})$ for every $1/2 < p < \infty$ and $0 < q \leq \infty$. Moreover, $U_n^{\theta_5} \rightarrow f$ a.e. as $n \rightarrow \infty$ and $V_{\mu}^{\theta_5} \rightarrow f$ a.e. as $\mu \rightarrow 0$. Note that the Riemann summation was considered in Bari [1], Zygmund [23], Gevorkyan [6, 7] and also in Weisz [20].

EXAMPLE 6. de La Vallée-Poussin summation. Let

$$\theta_6(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2 \\ -2|x| + 2 & \text{if } 1/2 < |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

One can show that $U_{2n+1}^{\theta_6} f = 2U_{2n+1}^{\theta_4} f - U_n^{\theta_4} f$ and since $\theta_6(x) = 2\theta_4(x) - \theta_4(2x)$, we have $|\hat{\theta}'_6(x)| \leq C/x^2$ (cf. Schipp and Bokor [11]). Hence we get again Theorems 3 and 4 and Corollary 1 for $N=0$. Note that we could generalize this summation if we take in the definition of θ_6 another number than $1/2$.

EXAMPLE 7. *Rogosinski summation.* Let

$$\theta_7(x) = \begin{cases} \cos \pi x/2 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

and

$$U_n^{\theta_7} f(x) := \sum_{k=-n}^n \cos\left(\frac{\pi |k|}{2(n+1)}\right) \hat{f}(k) e^{ikx}.$$

Since

$$\hat{\theta}_7(x) = \frac{\sin(x - \pi/2)}{2(x^2 - (\pi/2)^2)}$$

(see e.g. Schipp and Bokor [11]), we can verify that $|\hat{\theta}'_7(x)| \leq C/x^2$ and so we obtain Theorems 3 and 4 and Corollary 1 for $N=0$.

EXAMPLE 8. *Jackson-de La Vallée-Poussin summation.* Let

$$\theta_8(x) = \begin{cases} 1 - 3x^2/2 + 3|x|^3/4 & \text{if } |x| \leq 1 \\ (2 - |x|)^3/4 & \text{if } 1 < |x| \leq 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

One can find in Butzer and Nessel [3] that

$$\hat{\theta}_8(x) = \frac{3}{\sqrt{8\pi}} \left(\frac{\sin x/2}{x/2} \right)^4.$$

Therefore we can show by elementary computations that $|\hat{\theta}_8^{(i)}(x)| \leq C/x^4$ for $i=0, 1, 2, 3$. Consequently, we have the corresponding results with $N=2$.

EXAMPLE 9. *The Summation method of cardinal B-splines.* For $m \geq 2$ let

$$M_m(x) := \frac{1}{(m-1)!} \sum_{k=0}^l (-1)^k \binom{m}{k} (x-k)^{m-1} \\ (x \in [l, l+1), l=0, 1, \dots, m-1)$$

and

$$\theta_9(x) = \frac{M_m(m/2 + mx/2)}{M_m(m/2)}.$$

It is shown in Schipp and Bokor [11] that θ_9 is even and

$$\hat{\theta}_9(x) = \frac{1}{\pi m M_m(m/2)} \left(\frac{\sin x/m}{x/m} \right)^m.$$

It is easy to see that $|\hat{\theta}_9^{(i)}(x)| \leq C/x^m$ for $i=0, 1, \dots, m-1$. Thus Theorems 3 and 4 and Corollary 1 hold for $N=m-2$.

The next example satisfies the conditions of Theorem 2.

EXAMPLE 10. *Riesz summation.* Let

$$\theta_{10}(x) = \begin{cases} (1 - |x|^\gamma)^\alpha & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

for some $0 < \alpha \leq 1 \leq \gamma < \infty$. The Riesz operators are given by

$$U_n^{\theta_{10}} f(x) := \sum_{k=-n}^n \left(1 - \left| \frac{k}{n+1} \right|^\gamma \right)^\alpha \hat{f}(k) e^{ikx}.$$

We proved in [21] that

$$|\hat{\theta}_{10}(x)|, |\hat{\theta}'_{10}(x)| \leq C/x^{\alpha+1}$$

As $0 < \alpha \leq 1$, in Theorem 2 we have $N=0$. It is easy to see that $C/x^{\alpha+1} \in L_{p_0}[\varepsilon, \infty)$ if and only if $p_0 > 1/(\alpha+1)$ and $C/x^{\alpha+1} \in L_{p_0, \infty}[\varepsilon, \infty)$ if and only if $p_0 \geq 1/(\alpha+1)$. Consequently, (12) holds for $1/(\alpha+1) < p < \infty$ and (13) for $1/(\alpha+1) \leq p_0 < \infty$ and the endpoints are the same in Theorem 4 and Corollary 1.

6. θ -SUMMATION OF FOURIER TRANSFORMS

In this section we summarize briefly the above results for Fourier transforms. First we introduce the Hardy spaces on the real line.

The *Fourier transform* of a tempered distribution f is denoted by \hat{f} . The non-tangential maximal function of a tempered distribution is defined by

$$f^*(x) := \sup_{t>0} |(f * P_t)(x)|,$$

where

$$P_t(x) := \frac{ct}{t^2 + x^2} \quad (t > 0, x \in \mathbf{R})$$

is the non-periodic Poisson kernel.

The *Hardy–Lorentz space* $H_{p,q}(\mathbf{R})$ ($0 < p, q \leq \infty$) consists of all tempered distributions f for which

$$\|f\|_{H_{p,q}(\mathbf{R})} := \|f^*\|_{p,q} < \infty.$$

For a tempered distribution $f \in H_p(\mathbf{R})$ ($0 < p < \infty$) the *Hilbert transform* or the *conjugate distribution* \tilde{f} is defined by

$$\tilde{f} := f * \Phi,$$

where

$$\hat{\Phi}(u) = -i \operatorname{sign} u, \quad \Phi(x) = \frac{1}{\pi x}.$$

We remark that the analogues of (1), (2) and (3) and the analogues of Theorem A, B and C are true in this case (cf. Weisz [21] and the references there).

For $f \in L_p(\mathbf{R})$ ($1 \leq p \leq 2$) the θ -means are defined by

$$U_T^\theta f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta\left(\frac{t}{T}\right) \hat{f}(t) e^{ixt} dt = (f * K_T^\theta)(x),$$

where

$$K_T^\theta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta\left(\frac{t}{T}\right) e^{ixt} dt = T\hat{\theta}(Tx).$$

Thus the θ -means can be rewritten as

$$U_T^\theta f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) T\hat{\theta}(T(x-t)) dt.$$

We extend the definition of the θ -means to tempered distributions as follows:

$$U_T^\theta f := f * K_T^\theta \quad (T > 0).$$

One can show that $U_T^\theta f$ is well defined for all tempered distributions $f \in H_p(\mathbf{R})$ ($0 < p \leq \infty$) and for all functions $f \in L_p(\mathbf{R})$ ($1 \leq p \leq \infty$) (cf. Stein [15]). The definition of the *conjugate θ -means* is

$$\tilde{U}_T^\theta f := \tilde{f} * K_T^\theta \quad (T > 0).$$

The *maximal* and *conjugate maximal θ -operators* are introduced by

$$U_*^\theta f := \sup_{T>0} |U_T^\theta f| \quad \text{and} \quad \tilde{U}_*^\theta f := \sup_{T>0} |\tilde{U}_T^\theta f|,$$

respectively.

We can prove all the results of Section 4 also for tempered distributions and Fourier transforms and for the Hardy spaces $H_{p,q}(\mathbf{R})$. We do not formulate exactly the theorems and proofs, because they are almost the same as in Section 4.

THEOREM 5. *Theorems 1–4, Proposition 1 and Corollary 1 hold also for the operators U^θ acting on tempered distributions and defined in this section and for the Hardy spaces $H_{p,q}(\mathbf{R})$.*

Note that the applications of Section 5 are also special cases of the θ -summation of Fourier transforms, the details are left to the reader.

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